Chaotic dynamics of SU(2) gauge fields in the presence of static charges*

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Abstract

We have found in numerical simulations that the chaoticity of the classical hamiltonian lattice SU(2) gauge field system is reduced in the presence of static charges at the same total energy. The transition to non-chaotic behavior is rather sudden at a critical charge strength. The equipartition of chromoelectric and chromomagnetic energy takes place on a time scale essentially faster than the leading Lyapunov exponent.

1 Introduction

Knowledge of the mechanisms responsible for local equipartition of energy carried by non-abelian gauge fields and in particular knowledge of the speed by which the allowed phase space of possible field configurations is filled are important for the understanding of processes which lead to equilibrium in the very early universe and in relativistic heavy ion collisions. Prime examples for such processes are baryogenesis during the electroweak phase transition, the creation of primordial fluctuations in the density of galaxies, and the thermalization of excited quark matter in relativistic heavy ion collisions. Deterministic chaos of extended hamiltonian systems, like lattice models of non-abelian gauge theory are, is a universal mechanism of ergodization due to soft classical fields. Its main effect, the heating up of the system, is in addition to hard parton scattering.

In the recent past numerical studies of the real-time evolution of classical gauge field systems demonstrated chaotic behavior of randomly chosen

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configurations with chromomagnetic energy[1]. The maximal Lyapunov exponent (measured in lattice units) has been found to scale linearly with the energy of the system (also in appropriate units) making thus an extrapolation to the infinite dimensional field theory in the classical continuum limit possible. The linear scaling revealed that the maximum Lyapunov exponent survives in the continuum limit, so the real physical system has a finite entropy density generation rate. Abelian systems on the other hand do not scale linearly leading to vanishing entropy generation in the continuum limit. Also analytic studies of simplified subsystems, like the xy - model helped to understand the trajectory - defocusing property of the potential energy, i.e. the magnetic part of the gauge field interaction[2].

In order to understand the intriguing coincidence between the numerically found leading Lyapunov exponent and the gluon damping rate calculated in high-temperature perturbative gauge theory one has to assume that the energy is fully thermalized and the gauge field system evolves to a weakly interacting gas of quanta, the gluons. Some arguments in favor of a special physical mechanism based on color non-diagonal interaction between gluons behind this coincidence has been discussed in [3, 4].

We expect that the non-abelian gauge field system is in its magnetic sector the most chaotic. In realistic situations, in particular in ultrarelativistic heavy ion collisions there are, however, matter fields: mainly quarks but also clusters of several quarks representing a higher color charge. For some specific signals, expected to come from such reactions if quark - gluon matter has been formed, there are heavy color charges which generate strong chromoelectric fields in string - like configurations. As a matter of fact the formation and physical behavior of such strings play also a primary role in understanding the quark confinement mechanism; the very basis of the existence of hadrons.

It is therefore of genuine interest to investigate the effect of such chromoelectric strings spanned between static charges on the chaotic dynamics of random chromomagnetic gauge field configurations. Although speculative, it also cannot be excluded that we learn about the confinement mechanism by simulating the quantum mechanical ground state of QCD, i.e. the hadronic vacuum, with random classical chromomagnetic fields. Indeed a pair of in total neutral color charges, quantized or classical, connected by a single flux line can only be represented by a so called Wilson line, the object on which static quark confinement has ever been studied in four dimensional euclidean lattice gauge theory. A transition in the chaotic dynamical behavior of the classical gauge field system depending on the strength of this flux line, on the string constant, may therefore also be related to the confinement - deconfinement phase transition.

Even more speculative but still not totally negligible the idea that random classical field configurations numerically simulate quantum dynamics. In fact interference patterns similar to those observed in basic quantum mechanical two slit experiments can be created numerically by random discrete maps used in certain chaos games[5, 6, 7]. Whether the simulation reported in the present article has anything to do with such phenomena must be a subject of future considerations.

We organize this paper as follows. First the basic formuli for the classical hamiltonian SU(2) lattice gauge field theory are presented. Then the speed of energy sharing between chromomagnetic and chromoelectric degrees of freedom with and without static charges is discussed. Finally the effect of strong flux line initialization on the leading Lyapunov exponent of the chaotic gauge field system is described.

2 The lattice SU(2) model

In the hamiltonian lattice formulation of classical SU(2) gauge field theory, which we use to study its chaotic dynamics, the basic variables are group valued on each link of a N^3 cubic lattice[8, 9]

$$U_{x,i} = \exp\left(-\frac{i}{2}\tau^a g a A_i^a(x)\right). \tag{1}$$

In the SU(2) model, the smallest nonabelian gauge group, the τ^a-s are the Pauli matrices and the $A_i^a(x)$-s are corresponding components of the nonabelian vector potential. The subscript (x, i) identifies a general lattice position x in the interval $(0, N^3 - 1)$ and i the spatial direction of a link $i = 0, 1, 2$. Since we use periodic lattices this link variable representation is complete; the periodic boundary conditions are taken into account in an index table of neighboring sites in each of the 6 possible spatial directions. The parameters g and a are the bare coupling constant and the lattice spacing, respectively. In the classical time evolution actually all space- or timelike quantities can be measured in units of a and all types of physical energies in units of $g^2 a$.

While the vector potential is related to the basic link variable $U_{x,i}$ the magnetic field strength can be reconstructed from a product of U matrices along an elementary closed line, the plaquette:

$$U_P = U_{x,ij} = U_{x,i} \cdot U_{x+i,j} \cdot U_{x+i+j,-i} \cdot U_{x+j,-i}. \tag{2}$$

In the continuum limit $a \to 0$ the subleading term is proportional to the

chromomagnetic field

$$U_{x,ij} = \exp\left(-\frac{i}{2}\tau^a g^2 a \epsilon_{ijk} B^a_{x,k}\right). \tag{3}$$

Here ϵ_{ijk} is the totally antisymmetric three dimensional tensor. In practical calculations it is advantageous to use the complement link variable, $V_{x,i}$, instead of the plaquette matrix U_P. It is defined so, that the sum of the four plaquettes leaning on a given link gives exactly $U \cdot V^\dagger$.

The chromoelectric field is related to the time derivative of the basic link variable, $\dot{U}$,

$$E^a_{x,i} = -\frac{ia}{g^2}\mathrm{tr}\left(\tau^a \dot{U}_{x,i} U^\dagger_{x,i}\right). \tag{4}$$

An exactly good albeit not equivalent definition is

$$E^a_{x,i} = -\frac{ia}{g^2}\mathrm{tr}\left(\tau^a U^\dagger_{x,i} \dot{U}_{x,i}\right). \tag{5}$$

Again, in practical simulations it is more advantageous to use the canonical momentum $P = \dot{U}$ attached to each link instead. The Hamiltonian of the SU(2) lattice gauge theory is given by

$$H = \sum\left(\frac{1}{2}\langle P, P\rangle \; + \; 1 - \langle U, V\rangle\right) \tag{6}$$

in terms of link variables. The summation runs over all links. Here the scalar product of two group elements is given by

$$\langle A, B\rangle = \frac{1}{2}\mathrm{tr}(AB^\dagger) \tag{7}$$

in the matrix representation and by the usual scalar product of two real four dimensional vectors in the quaternion representation. The dynamics generated by this Hamiltonian conserves the unitarity of the group elements $\det U =< U, U >= 1$, the orthogonality of the basic variables and the corresponding canonical conjugates $\langle P, U\rangle = 0$, and commutes with the classical Gauss' law. The latter describes the conservation of color charge locally, i.e. conserves the following quantity calculated on each lattice sites:

$$\Gamma = \sum_{+} PU^\dagger - \sum_{-} U^\dagger P, \tag{8}$$

where the "+" summation runs over links originating in the site and the "-" over those links which end at the site.

For the numerical solution of the equations of motion we applied a mixed explicit - implicit algorithm which exactly fulfills the constraint of

the Noether charge conservation also for discrete timesteps. A derivation of this property is given in [10]. The implicit algorithm,

$$\begin{aligned} U' &= U + (P' - \epsilon U) \\ P' &= P + (V - \mu U + \epsilon P') \end{aligned} \tag{9}$$

with

$$\epsilon = \langle U, P' \rangle \qquad \text{and} \qquad \mu = \langle V, U \rangle + \langle P', P' \rangle \,, \tag{10}$$

where the primed quantities denote the updated (new) values and the elementary timestep is included in the usage of $P = dt\dot{U}$, can be resolved into explicit equations. Eventually it can be comprised into the following algorithm

$$\begin{aligned} V_T = V - \langle U, V \rangle \, U \qquad & \qquad \tilde{P} = P + V_T \\ c \quad = \quad & \sqrt{1 + \left\langle \tilde{P}, \tilde{P} \right\rangle} \\ P' = (\tilde{P} + U)/c - cU \qquad & \qquad U' = cU + P'. \end{aligned} \tag{11}$$

It is important to recalculate implicit recursion formuli analytically, because explicit equations are not only faster on computer but they also conserve the Noether charge (in our case Gauss' law) exactly, i.e. up to the highest precision used in the computer code independently of the discrete time step. An implicit algorithm, like (9), would have been solved iteratively so the exactness of the charge conservation depended on the iteration depth.

The explicit numerical algorithm we use conserves Gauss' law, i.e. the lattice covariant divergence of the nonabelian electric field (eq.8). The only remaining task is to fulfill it initially. For a purely chromomagnetic configuration, investigated dynamically up to now, it is trivial to set the color charge Γ on each lattice site to zero by initializing $P = 0$ on all links. This state can also be viewed as a "chromoelectric vacuum".

To describe localized, i.e. on the lattice pointlike, color charges so that the total configuration is color neutral, is far from trivial due to the nonabelian nature of Gauss' law in the SU(2) gauge theory. In fact no general nonperturbative solution of the classical Gauss' law is known for arbitrary color charge density. We follow here, however, a simpler formulation of the problem: how to initialize the variables P on the links once a U - configuration is given, the total system is color neutral and the charges are localized. Our construction principle is as follows.

1. We generate a U - configuration randomly (microcanonically) which represents purely chromomagnetic energy.

2. We prescribe a flux line by a sequence of $3N$ neighboring lattice sites. In particular we choose a diagonal staircase string across the N^3 cube.
3. On the first link of this flux line $P_1 = QU_1$ is initialized.
4. In order not to have any charge in between we initialize $P_n = U_{n-1}^\dagger P_{n-1} U_n$ for $(1 < n < 3N)$ recursively along the prescribed flux line.
5. Arriving at the end of the line the last site has a single incoming link with nonzero P on it. Its uncompensated charge is $\Gamma = -F^\dagger Q F$ where F is the path ordered product of basic U variables along the flux line (a Wilson line):

$$F = \prod_n U_n. \tag{12}$$

6. Finally in order to ensure color neutrality of the system we need $Q - F^\dagger Q F = 0$ for the total charge and tr $Q = 0$ for ensuring orthogonality of P and U everywhere. The only solution to this problem is

$$Q = \frac{q}{2}\left(F^\dagger - F\right), \tag{13}$$

where the real parameter q describes the strength of the flux line or equivalently the magnitude of the end charges.

3 Less chaos due to flux lines

We measured the leading Lyapunov exponent as the logarithmic time-derivative of the gauge invariant distance, $D(t)$, between two initially adjacent field configurations

$$h = \frac{d}{dt} \ln D(t). \tag{14}$$

This coincides with the original definition of the Lyapunov exponent whenever $\ln D(t)$ linearly grows. The numerical value of h was fitted in such time-intervals only using a time step $a = 0.005$ equal to the lattice spacing for the sake of simplicity. The gauge invariant distance of two field configurations is defined by

$$D(t) = \sum \left| \langle U(t), V(t) \rangle \; - \; \langle U^*(t), V^*(t) \rangle \right|, \tag{15}$$

where the superscript * denotes a configuration only slightly different from the unstared one initially and the summation runs over all links. This

definition of gauge invariant distance differs from the one used earlier [1] - defined using the plaquette sum - only in the actual value of the distance but not in its growth rate from which we obtain the leading Lyapunov exponent.

The initially small distance between two trial configurations we produce by rotating randomly all link quaternions, U, by an SU(2) group element near to unity. Although – in principle – we would have to average the value of the leading exponent obtained this way over the phase space, the ergodizing property of the chaotic evolution ensures that both trajectories sample almost the whole available phase space in a single long term run.

Fig.1 shows the evolution of the logarithmic distance, $\ln D(t)$, for an $N = 6$ cubic lattice with the total scaled energy per link of $g^2aE = 2.00$. The different curves counted from above correspond to charges $q = 12.10$, 0.0, 8.70, 9.50 at the ends of the flux line and to respectively reduced random magnetic backgrounds, belonging to an independent random choice of SU(2) group elements with the main group angle covering uniformly 45%, 100%, 35% and 15% of the interval $[0, 2\pi]$.

A tendency of faster initial oscillations in the divergence of the two parting configurations with increasing strength of the flux line can be observed. It is followed by linear epoches with a reduced slope, i.e. reduced chaotic dynamics. Correspondingly the entropy generation is also reduced.

The equipartition of the chromoelectric and chromomagnetic energy is essentially faster than the entropy generation rate set by the leading Lyapunov exponent. Of course the partition of the total lattice energy (not counting the energy of the static charge) develops already in the linear approximation without any chaotic behavior. Its time scale is determined by the highest oscillation frequency, $\sim \pi/a$, of the lattice system and after a few oscillation periods the equipartition is completed.

A systematic study of the leading Lyapunov exponents as a function of the charge strength at the ends of the initial flux line reveals an S-like structure known from first order phase transitions (cf. Fig.2). The slight initial rise of the upper branch may be due to numerical uncertainties; the $N = 6$ system is relatively small in linear size, although the allowed phase space is 3239 - dimensional. The qualitative feature of a microcanonical simulation of a phase transition between weak and strong charges is, however, inspiring.

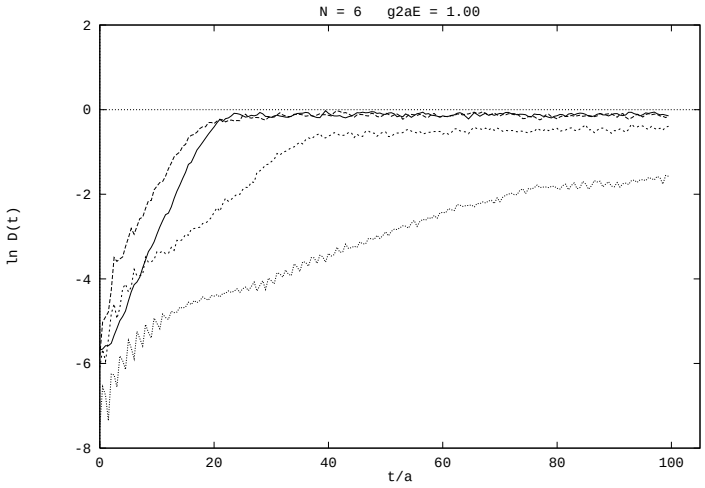

Figure 1: The evolution of the gauge invariant distance between two initially adjacent configurations with different static charges at the ends of a diagonal flux line in a 6^3 cubic lattice. The total energy per link per color is $g^2aE = 1.00$ for all curves.

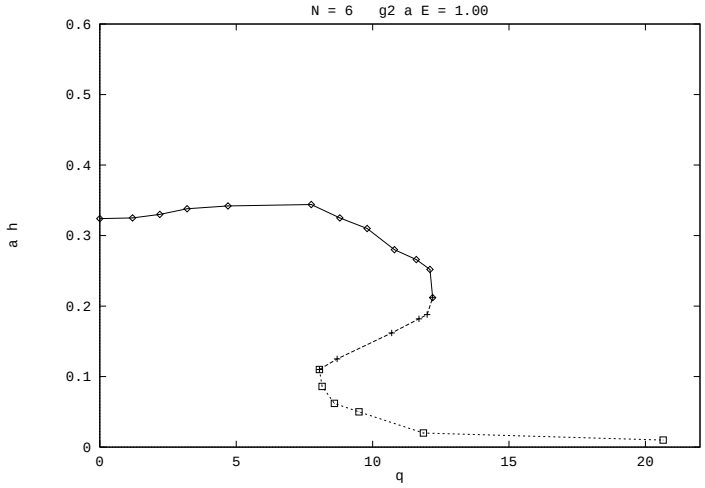

Figure 2: The leading Lyapunov exponent as function of the static charges positioned at the ends of a diagonal flux line.

In conclusion these simulations have shown that the chaotic dynamics of classical SU(2) Yang-Mills fields survives in the presence of weak static charges and a thin flux line of chromoelectric energy spanned between them. In order to reduce the chaos essentially a strong (in the order of magnitude of 10 in scaled units) charge is necessary. This reduction occurs then suddenly at a given critical strength and shows a hysteresis – like in case of a ferromagnetic phase transition.

With respect to relativistic heavy ion collisions this result leaves room for the hope that chromoelectric strings (color ropes) formed initially share their energy with random chromomagnetic fields – which bundle them together – and therefore the soft collective chaotic dynamics of these fields is not suppressed essentially. This chaotic dynamics then contributes to the thermalization of primordial quark matter helping it to evolve towards a quark - gluon plasma.

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